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DISCRETE CIRCLE ACTIONS: A NOTE USING NON-STANDARD ANALYSIS

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Dedicated to the memory of Alex Zabrodsky

ABSTRACT

In this note we prove that the existence of effective uniformly Lipschitz $\mathbb{Q}/\mathbb{Z}$ actions on manifolds (and other spaces) follows from the existence of such Z_n actions. The method of approach is non-standard analysis with all non-trivial transformation group theoretical information concentrated in Newman's theorem; this results in a completely elementary argument. We give examples showing that, in contrast, there are spaces with no S^1 effective actions despite Z_n and hence $\mathbb{Q}/\mathbb{Z}$ effective actions.

Introduction and statement of results

This note shows how one can use the insights of non standard analysis to give brief elementary proofs regarding the existence of group actions on certain manifolds. In general, the goal would be to study questions about compact Lie group actions

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in terms of actions of finite subgroups. (We concentrate on the first interesting case, that of the circle.) While these can be studied by purely topological means in cases where there is simple orbit structure (see [W] for some discussion of free actions), here metric hypotheses are exploited.

Such Lie groups can be understood by the tools of non-standard analysis (see [H] for a proof of Hilbert's 5th problem in this mode) and the idea of somehow understanding a structure by its finite components is a standard theme in non-standard applications ([Ro], [M]). Here we will restrict ourselves to a quick proof of the theorem stated in the abstract and then give some indications of the difficulties that arise in extending the method, e.g. for *free* actions or for S^1 actions. In particular, we give example of spaces and manifolds with free $\mathbb{Q}/\mathbb{Z}$ action but no continuous S^1 action (with the topology of the circle and the manifold).

One way of looking at the results is as a kind of "transfer" property like the Lefschetz principle in algebraic geometry. This ([R1] and [M]) suggested the use of mathematical logic, which is realized here in the ideas and framework of non-standard analysis. On the one hand, this is a reasonably sophisticated tool of applied logic ([R2], [D]) but its use makes strong topological and group-theoretic tools unnecessary. Moreover, the tools from non-standard analysis used are fairly simple ones, so the proofs here are almost self-contained. (We include a micro-introduction to non-standard analysis.) As a result, for the work presented here, all the arguments are completely natural and elementary; it turns out that all the transformation group-topological information needed can be isolated in a version of Newman's theorem.

The idea for this paper was first conceived of during a discussion following a lecture by W. on [W] that M. happened to attend. We mention this because Alex Zabrodsky also was present and in his indubitable manner immediately saw both the mathematical point and the humor in using logic on this problem. Accordingly, we dedicate this work to his memory.

NEWMAN'S THEOREM (Here is a version appropriate for our purposes. (See [Br] for a proof and other versions.)):

Suppose M is a connected manifold with a metric. Then there is an $\epsilon > 0$ such that, for every effective action of a compact Lie group on M, there exists an orbit of diameter larger than ϵ.

NON-STANDARD ANALYSIS (We give a telegraphic account with a couple of deliberate mis-statements. Most of the subtleties will not be important for the understanding of this paper. For a more complete treatment see [Ro] or [D].):

Let V be any usual model of mathematics (e.g. of ZFC set theory). Let $^*V$ be an extension of V such that:

(i) *We may assume a mapping $^*\colon V \longrightarrow {}^*V$ (denote $^*(a)$ as $^*a$). This mapping is elementary in the sense that any sentence $\psi(a_1, \dots, a_n)$ is true in V about $a_1, \dots, a_n$ if and only if $\psi(^*a_1, \dots, {}^*a_n)$ is true in $^*V$ about $^*a_1, \dots, {}^*a_n$.*

(ii) *There is a second mapping $e\colon V \longrightarrow {}^*V$ such that $e(a) = \{^*b \mid b \in a\} \subset^* a$. This is called the standard part of $^*a$. Usually we omit the e and write simply $a \subseteq^* a$.*

(iii) *If a is infinite then $^*a \supset a$ and $a \neq a$. If a is finite then (for the purposes of this paper one may assume) $^*a = a$.[1] For example, one may take ^{*1} as 1, ^{*2} as 2, ... etc., but $^*\omega \supsetneqq \omega$. Moreover, $^*\langle \omega, < \rangle \supsetneqq \langle \omega, < \rangle$, i.e. there are elements in $^*\omega \setminus \omega$ which in $^*V$ are larger than $1, 2, 3, \dots$. (These are called* **infinite** *natural numbers.) It is possible to choose $^*V$ so that there is a $\gamma \in^* \omega \setminus \omega$ such that $n \mid \gamma$ for all $n \in \omega$.*

By property (i) recall that all facts about ω in V remain (when properly interpreted) true about $^*\omega$ in $^*V$. Thus, for example, every subset of $^*\omega$ in $^*V$ has a first element. Since the collection of infinite members of $^*\omega$ has no such first element, this is a subset of $^*\omega$ not in $^*V$. (This is an example of an **external** set.)

Similar **transfer** properties follow for all mathematical structures. Thus there is a group $^*Z_n$ for each natural number n, and since these are finite structures $^*Z_n \approx Z_n$. However, in $^*V$, there is a group Z_γ for all infinite γ as well.

Given a topological space $\langle M, \tau \rangle$, there is a corresponding space $^*\langle M, \tau \rangle$ in $^*V$. As before M may be taken as a subset of $^*M$. The (usual) subtle point is that while "$^*\langle M, \tau \rangle$ is a topological space" holds in $^*V$, actually $^*\langle M, \tau \rangle$ is not a true topological space because $^*\tau$ will not be closed under union over an *external* collection of sets. Note that, on the other hand, a *group is a group.

$^*\tau$ will consist of more than $\{^*A \mid A \in \tau\}$ (if τ is infinite). However, if Γ is the set of standard neighborhoods of a point $a \in M$, we will say that $b \in^* V$ is infinitesmally close to a ($b \approx a$) or b is in the monad of a ($b \in \mu(a)$), if $b \in \bigcap^*_{A \in \Gamma} A$. For example, if M is a metric space, then $\mu(a) = \{b \in^* V \mid |b - a| < 1/n$ for all

1 Actually, there is some deliberate fudging here. See [D] or [R2].

$n \in \omega\}$.

In general $\mu(a) \neq \emptyset$. However, a space M is Hausdorff, if and only if for all $a, a' \in M, a \neq a', \mu(a) \cap \mu(a') = \emptyset$. It is natural to ask which points in $^*M$ are in the monad of some member of M. M is *compact* if and only if every element of $^*M$ is in the monad of some element of M. Thus, for compact Hausdorff spaces, there is a unique map $^*M \longrightarrow M$ denoted sp(a) or $^\circ a$, sending any element of $^*M$ to the infinitesimally close member of M.

Statement of results and proofs

STATEMENT OF RESULTS.

Definition: Let $\langle M, d\rangle$ be a compact or locally compact metric space, G a group, and K a positive real number. A group action of G on $\langle M, d\rangle$ is said to be K-**Lipschitz** or **quasi-isometric** if for each $g \in G$, $d(g(a), g(b)) \leq Kd(a, b)$.

MAIN THEOREM: *Let M be as in Newman's theorem and suppose that for each n there is an effective K-Lipschitz action of Z_n on M. Then there is an effective K-Lipschitz $\mathbb{Q}/\mathbb{Z}$ action on M.*

Note the following immediate corollary:

COROLLARY: *Suppose that there is a Z_n action by isometries on M for each natural number n. Then there is a $\mathbb{Q}/\mathbb{Z}$ action by isometries on M.*

Note: 1. This corollary can be proven as well using Hilbert's 5th problem and general facts about compact Lie groups. That proof would also show the existence of a circle action. The topological approach has only been successful for free $\mathbb{Q}/\mathbb{Z}$ actions. (See [W].)

2. There are other versions of Newman's theorem than that for connected manifolds with metric [Br]. The main theorem will apply to any such space; for example, it applies to polyhedral and locally polyhedral spaces.

3. See below for examples of spaces which cannot be acted on effectively by S^1 with the topology of the reals even though there are $\mathbb{Q}/\mathbb{Z}$ actions.

4. It would be interesting to come up with conditions guaranteeing such an effective S^1 action. The following simply states the criterion for extending a $\mathbb{Q}/\mathbb{Z}$ action to a circle action.

Definition: A $\mathbb{Q}/\mathbb{Z}$ action is **coherent** if each function $a\colon \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$ determined by $q(a)$ for each $q \in \mathbb{Q}/\mathbb{Z}$ is continuous in $\mathbb{Q}/\mathbb{Z}$ (with the subset topology inherited from the reals).

PROPOSITION: *There is an S^1 action (with the topology of the reals) on M if and only if there is a $\mathbb{Q}/\mathbb{Z}$ "coherent" action on M.*

(This has an obvious local (i.e. Z_n) version which is left as an exercise to the reader. A condition on Z_n actions which do not seem a priori to generate a "coherent" $\mathbb{Q}/\mathbb{Z}$ action would be desirable.)

In the following, we assume once and for all that ${}^*V$ is an appropriate nonstandard universe.

Proof of Theorem: For each n, there is a Z_n acting effectively on M. Hence, by property (i) "for every natural number n, ${}^*Z_n$ acts effectively on ${}^*M$" is true in ${}^*V$. Checking the meaning of this (e.g. Z_γ is a group and determines an action on the *set* ${}^*M$,[2] *effectiveness means effectiveness, ${}^*Z_n = Z_n$ for finite n) and choosing $\gamma \in^* \omega - \omega$ such that $n|\,\gamma$ for all $n \in \omega$, we have an effective Z_γ action on ${}^*M$.

Since M is compact, the standard part map is defined from ${}^*M$ onto M, hence one has the following diagram which determines, for each element of Z_γ, a map from M to M.

$$\begin{tikzcd}
{}^*M \arrow[r, "z\in Z_\gamma"] & {}^*M \arrow[d, "sp"] \\
M \arrow[u, hook] \arrow[r, dashed] & M
\end{tikzcd}$$

Since $n|\gamma$, we have $Z_n \subset Z_\gamma$ for each n. Identifying $\mathbb{Q}/\mathbb{Z}$ with $\bigcup_{n\in\omega} Z_n$, one has for each element of $\mathbb{Q}/\mathbb{Z}$ a map from M to M.

We now have to show that $\mathbb{Q}/\mathbb{Z}$ is an action and that it is effective. Actually even Z_γ acts on M, although it may not be effective since an element of ${}^*M$ may be sent to a *different* element of ${}^*M$ in the *same* monad.

2 Recall that ${}^*\langle M,\tau\rangle$ is not really a topological space.

(i) Z_γ is an action. This follows from the K-Lipschitz conditions. That is, since each Z_n is K-Lipschitz on M so is Z_γ on ${}^*M$.

Let $z \in Z_\gamma$. For the moment denote the mapping determined by the Z_γ action on ${}^*M$ as $z(a)$ while the mapping determined by the action on M will be denoted by $\bar{z}(a)$. Thus $\bar{z}(a) = {}^\circ(z(a))$ for $z \in Z_\gamma$, $a \in M$.

$$(\bar{z}_1\bar{z}_2)(a) = \overline{z_1 z_2}(a) = {}^\circ((z_1 z_2)(a)) = (\text{since } Z_\gamma \text{ acts on } {}^*M) = {}^\circ(z_1(z_2(a))).$$

Now by the K-Lipschitz property, we have $a \approx a' \Rightarrow z(a) \approx z(a')$, hence ${}^\circ(z_1(z_2(a))) = {}^\circ(z_1({}^\circ z_2(a))) = \bar{z}_1(\bar{z}_2(a))$. ∎

(ii) $\mathbb{Q}/\mathbb{Z}$ acts effectively on M. This follows from the version of the theorem of Newman stated in the introduction. Recall that associated with M there is a real number $\epsilon > 0$ such that each non-identity mapping determined by any effective action on M has an orbit of diameter at least ϵ. (Formally, the theorem states that there is a mapping with this property, but the above statement follows immediately by considering the subgroup generated by any non-identity mapping in the action.)

Since Z_γ acts effectively on ${}^*M$, Z_n also acts effectively on ${}^*M$ under the inherited action. Hence by the "transfer"[3] of Newman's theorem to ${}^*V$, each non-identity element of Z_n determines some orbit with diameter at least ϵ.

Now let q be any element of Z_n other than the identity and consider Z_n as acting with the action it inherits as a subgroup from Z_γ. Then, just restating the above, "there is an element of ${}^*M$ which, when acted upon by q repeatedly has an orbit of diameter larger than ϵ." Then by "transfer" again, this is true in V, i.e. "there is an element of M which when acted on repeatedly by $\bar{q}$ has an orbit of diameter larger than ϵ." This means that the induced map $\bar{q}$ is not the identity; hence Z_n and therefore $\mathbb{Q}/\mathbb{Z}$ acts effectively on M. ∎

COUNTER EXAMPLES, EXTENSIONS AND OTHER COMMENTS. This particular theorem was straightforward because all of the group-topological information needed was already contained in Newman's theorem. We consider some possible extensions: (1) working with non-metric spaces; (2) replacing $\mathbb{Q}/\mathbb{Z}$ actions with S^1 actions in the conclusion to the theorem.

3 I.e. the use of property (i) of non-standard analysis.

(1) Non-metric spaces. The above proof can be made to work without metricity and only assuming local compactness. One passes from local compactness to compactess by using the one-point compactification. The metricity can be dropped since there is a version of Newman's theorem for compact (or locally compact) spaces [Br 9.4]. This version states that there is a finite cover of the space such that any action must have an orbit that does not stay within one part of the cover. Since the covering is finite, one can name the parts and the part of the proof showing effectiveness will go through. However, to show that an action is defined, a non-metric version of the K-Lipschitz condition must be assumed.

(2) S^1 actions. One can ask about extending the action to S^1 actions (with the circle topology).

Here we give an example of a manifold which admits a free $\mathbb{Q}/\mathbb{Z}$ action but does not admit a fixed point free S^1 action (with the continuous topology on the circle). The method is to compare p-adic and rational homotopy types. Consider the 3-connected 8-manifold whose quadratic form is something like E_8+(hyperbolic). This is p-adically equivalent to the sum of 5 hyperbolics, i.e. the manifold $5(S^5 \times S^5)$. We cannot build our desired actions on these manifolds for Euler characteristic reasons, so connect sum each of these with $6(S^3 \times S^5)$. It is not hard to show that $5(S^5 \times S^5)\#6(S^3 \times S^5)$ has a free S^1 action. Since $X = (E_8 + (\text{hyperbolic}))\#6(S^3 \times S^5)$ is p-adically equivalent to this manifold, a direct modification of the method of [CW] using Sullivan's arithmetic square in place of the localization square produces free Z_{p^n} actions on the homotopy type of X for all p and n. Unfortunately the technology in [W] does not suffice to build these actions on X. However, after crossing with $\mathbb{R}^3$, the obstructions to doing this vanish [W]. This space cannot admit a (fixed point) free S^1 action, because if it did, the quotient would be a (rational) Poincaré space, and the mapping cylinder of the quotient map would be a (rational) nullcobordism, contradicting the fact that $\text{sign}(X) = 8$.

Remark: It seems certain that such an example exists on a compact manifold; indeed, it is likely that it is not necessary to cross with $\mathbb{R}^3$. Also there should exist examples where one can produce free smooth $\mathbb{Q}/\mathbb{Z}$ actions (or at least Z_n actions, for some finite n) by the method of [LR] on manifolds which cannot admit any smooth S^1 action for $\hat{A}$-genus reasons.

While the above example rules out a general transfer result, one can still ask about S^1 actions with the discrete topology. To use the methods of the first theorem, a stronger hypothesis on the actions is needed.

Definition: An action G acts ϵ-effectively on a manifold M if for every $g \in G$ there is a $m \in M$ such that the orbit of m determined by $\langle g \rangle$ has diameter at least ϵ.

An action acts strongly ϵ-effectively on a manifold with metric if for every g there is an $m \in M$ such that $|m^g - m| > \epsilon$.

So one can restate Newman's theorem by

Fact: Every effective action on a connected manifold with metric is ϵ-effective for some $\epsilon > 0$. (Newman's result says that ϵ depends only on the manifold.)

THEOREM: *If M is a manifold with strongly ϵ effective Z_n actions for each n then there is an S^1 strongly ϵ effective action with the discrete topology.*

Proof: The strong ϵ effectiveness is needed to keep the non-standard points from being trivial ones. Once this is assumed, it is routine by the techniques of non-standard analysis to lift from $\mathbb{Q}/\mathbb{Z}$ to S^1.

The example above shows that this last result cannot be extended to the circle with the usual topology. However, if one adds in addition that the Z_n actions fit in "coherently" with each other as in the second corollary above, then it is direct that there is an S^1 action with the usual topology.

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